MT 2 Solns

1) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{3x^2+y^2}$.

↳ This is "Top Heavy" ~ $\frac{r^3}{r^2}$ so we suspect limit exists, go polar.

$$\lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2(3\cos^2\theta+\sin^2\theta)} = \lim_{r\to 0} \frac{r\cos^3\theta}{1+2\cos^2\theta}$$ is our limit in polar.

We need to get independence of θ → set up squeeze thm,

- $\frac{r\cos^3\theta}{1+2\cos^2\theta} \leq \frac{r\cdot 1}{1}$ since $\cos^3\theta \leq 1$ and $1+2\cos^2\theta \geq 1$.
- $\frac{r\cos^3\theta}{1+2\cos^2\theta} \geq \frac{r\cdot(-1)}{1}$ since $\cos^3\theta \geq -1$ and $1+2\cos^2\theta \geq 1$.

Therefore, $-r \leq \frac{r\cos^3\theta}{1+2\cos^2\theta} \leq r$.

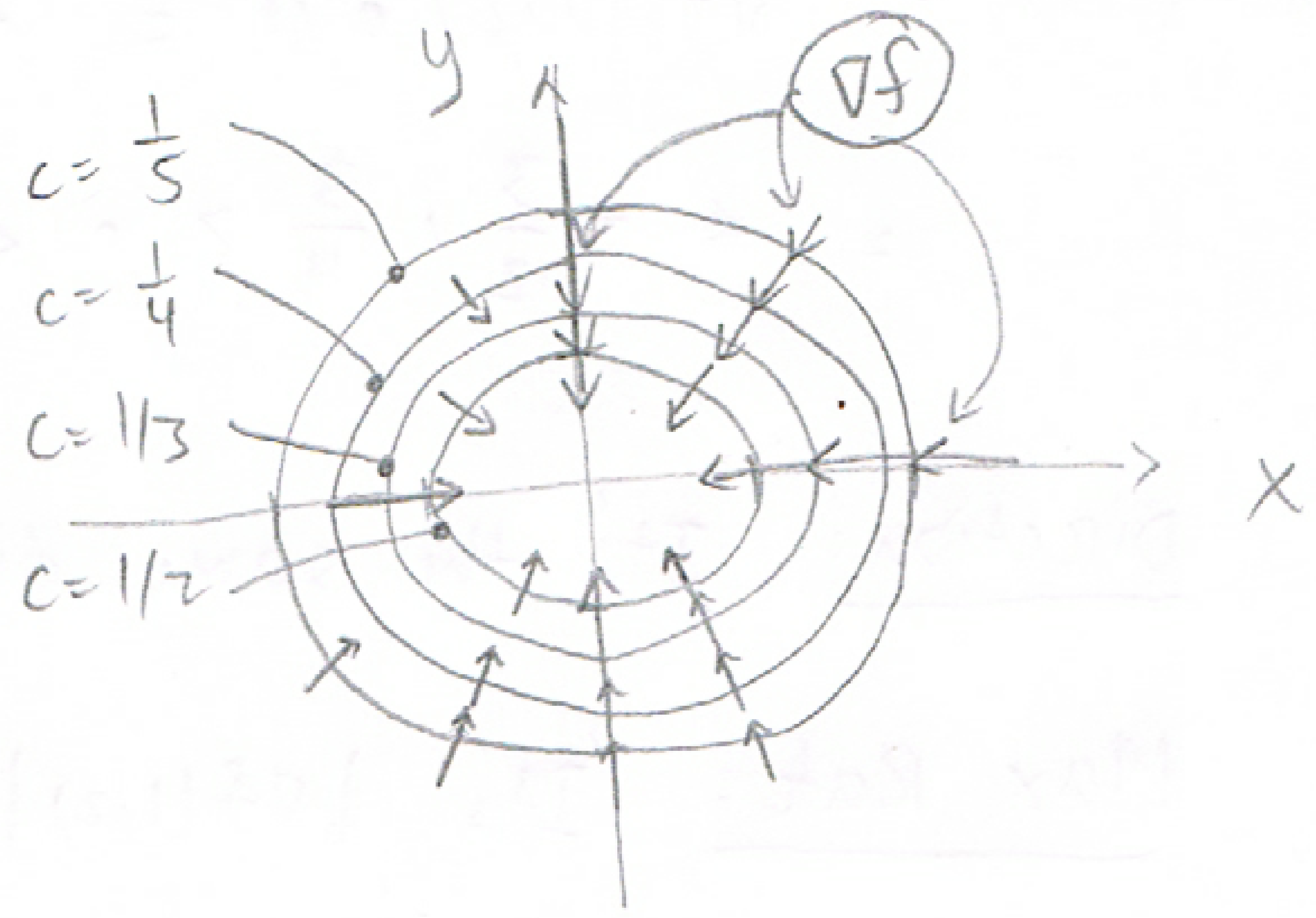
We see that $\lim_{r\to 0}(-r) = \lim_{r\to 0} r = 0$, so $\lim_{r\to 0}\frac{r\cos^3\theta}{1+2\cos^2\theta} = 0$, limit exists and equals 0 by Squeeze Thm.

b. $f(x,y) = c \rightsquigarrow \frac{1}{1+x^2+y^2} = c$; $1+x^2+y^2 = \frac{1}{c}$.

So our curves are $x^2+y^2 = \frac{1}{c}-1$. since $c = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, $\frac{1}{c} = 2,3,4,5$,

we have $x^2+y^2 = 4,3,2,1$ circles.

c. We expect the origin $(0,0)$ to be a critical pt as level curves converge to it. We infer it's a local max since ∇f points in direction of steepest ascent / increase.



2) a. We see $x^2+z^2=4$ when $x=2\sin t$, $z=2\cos t$ from the curve $r(t)$.

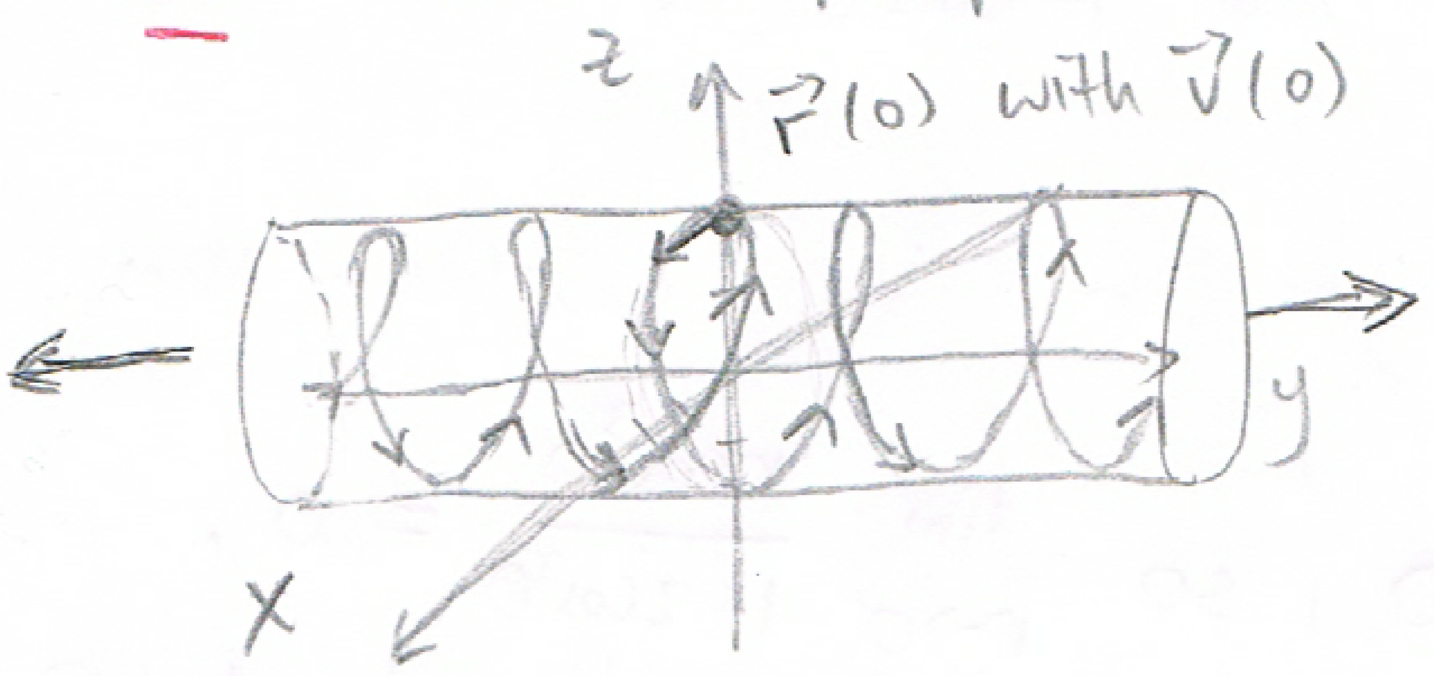
↳ This is an eqn. of a cylinder (radius 2) and the curve lies on the cylinder since any pt $(2\sin t, t^3, 2\cos t)$ satisfies the cylinder eqn because the y-var is free for cylinder.

b. $\vec{v}(0) = \vec{r}\,'(0) = \langle 2\cos t, 3t^2, -2\sin t\rangle \Big|_{t=0} = \langle 2,0,0\rangle.$

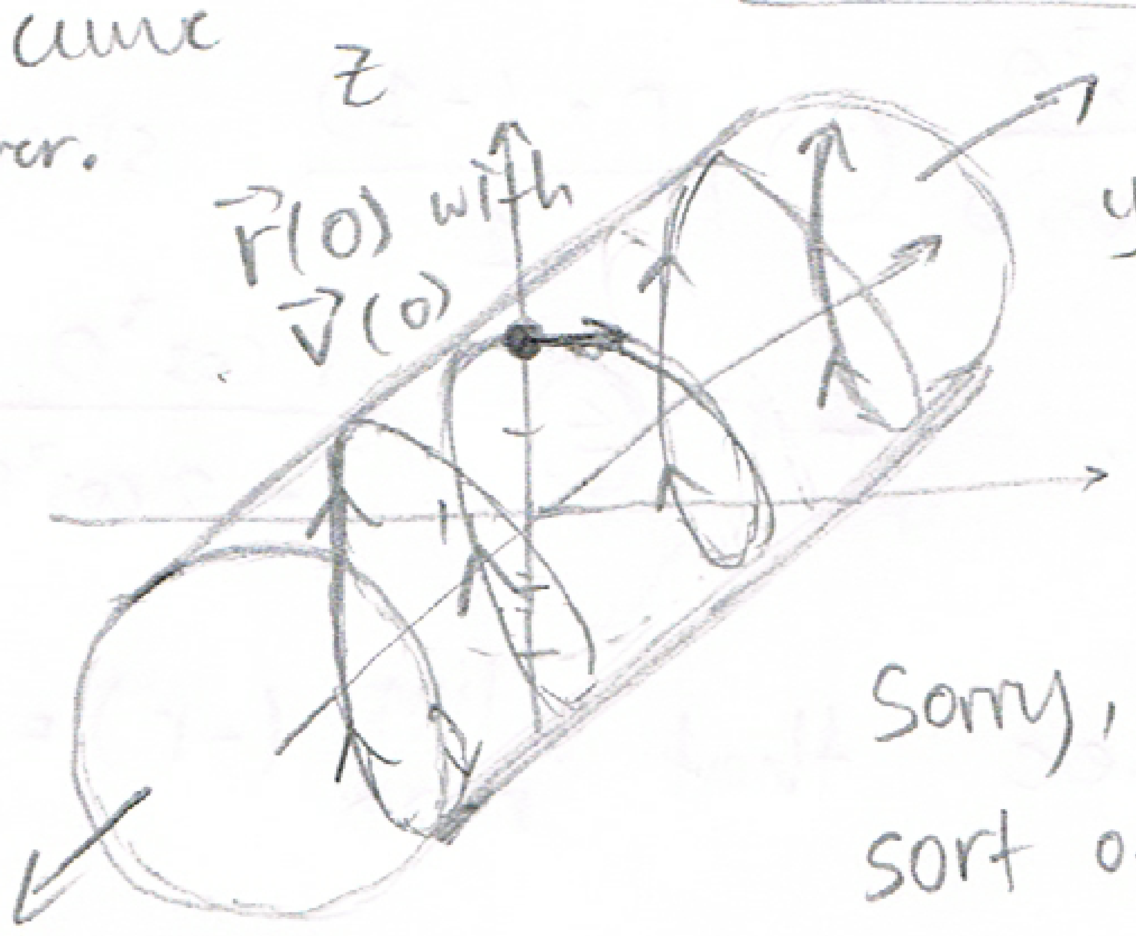
$\vec{r}(0) = \langle 0,0,2\rangle$

So, our tangent line at $\vec{r}(0)$ can be written as $\boxed{\begin{aligned} x &= 0+2t \\ y &= 0+0t \\ z &= 2+0t \end{aligned}}$

c. Here's 2 perspectives, note cylinder & curve go on forever.



and



Sorry, this one's sort of bad...

3) a. 1^{st}, $\langle 3,4\rangle$ direction is $\hat{u} = \dfrac{\langle 3,4\rangle}{\sqrt{3^2+4^2}} = \langle \frac{3}{5}, \frac{4}{5}\rangle.$

Now, $\nabla f(x,y) = \langle \frac{1}{y} - \frac{y}{x^2}, -\frac{x}{y^2}+\frac{1}{x}\rangle$. So,

$D_u f(1,2) = \nabla f(1,2)\cdot\hat{u} = \langle \frac{1}{2}-2, -\frac{1}{4}+1\rangle\cdot\hat{u}$

$= \langle -\frac{3}{2}, \frac{3}{4}\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = -\frac{9}{10} + \frac{3}{5} = \boxed{-\frac{3}{10}}$

b. Direction: It's the same direction as $\nabla f(1,2)$; recopy it, $\boxed{\langle -\frac{3}{2}, \frac{3}{4}\rangle.}$

Max Rate: It's $|\nabla f(1,2)|$, so $|\langle -\frac{3}{2}, \frac{3}{4}\rangle| = \left(\frac{9}{4}+\frac{9}{16}\right)^{1/2} = \left(\frac{36+9}{16}\right)^{1/2}$

↳ The max rate is $\boxed{\dfrac{\sqrt{45}}{4}}$

c. Near $(1,2)$, we have $f(x,y) \approx f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$.

Plug in $(1.02, 1.96)$, $f(1.02, 1.98) \approx \frac{5}{2} - \frac{3}{2}(0.02) + \frac{3}{4}(-0.04)$

$= 2.5 - 0.03 - 0.03 = \boxed{2.44}$

4) a. Plug in p, into level curve; $1^4+1^3+1^4+1^3=4$ ✓

b. We know $\frac{dy}{dx} = - \frac{F_x}{F_y}$ when $F(x,y) = x^4+x^3+y^4+y^3$ from level curve

$F_x = 4x^3+3x^2$; $F_y = 4y^3+3y^2$ $\Rightarrow$ $\boxed{\frac{dy}{dx} = - \frac{4x^3+3x^2}{4y^3+3y^2}}$

Now at $p=(1,1)$, $\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{7}{7} = \boxed{-1}$

c. Using point-slope, at $p=(1,1)$ with $m=-1$, $\underline{y-1=-1(x-1)}$

in other words, $\underline{y = 2-x}$.

d. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(- \frac{4x^3+3x^2}{4y^3+3y^2}\right)$, Be careful, y is a function of x!

$$= - \left[\frac{(12x^2+6x)(4y^3+3y^2) - (4x^3+3x^2)(12y^2+6y)\frac{dy}{dx}}{(4y^3+3y^2)^2}\right].$$

$\frac{dy}{dx} = -1$ at p ✓

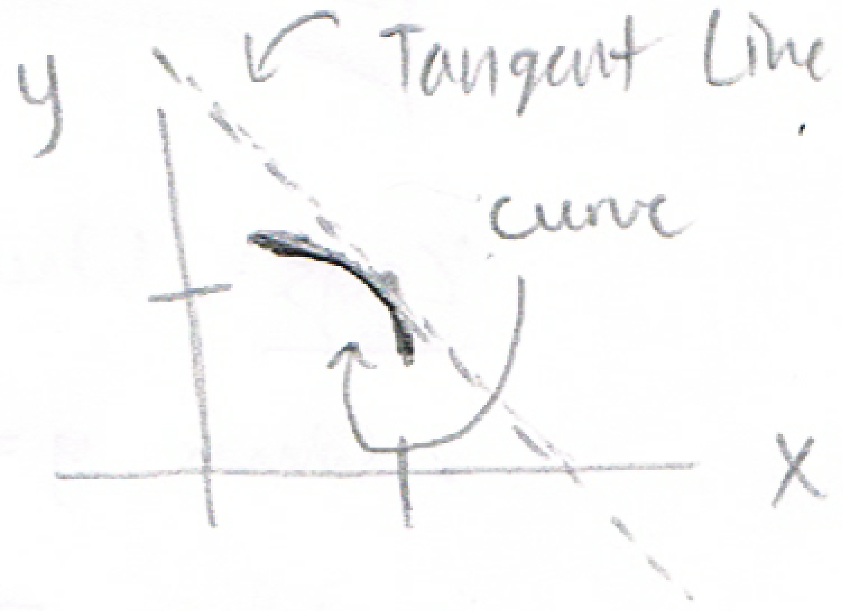
At (1,1), $\frac{d^2y}{dx^2} = - \left[\frac{(18)(7) - (7)(18)(-1)}{(7)^2}\right]$

$$= - \left[\frac{18+18}{7}\right] = \boxed{-\frac{36}{7}}$$

We see $\left.\frac{d^2y}{dx^2}\right|_{(1,1)} < 0$ and we have the tangent line from (c), so near $p=(1,1)$

(concave down) ("frowning").

The curve looks like



5) a. We need ∇f, $\nabla f(x,y) = \langle 2x-3y-8,\ 2y-3x+7 \rangle$.

1st for $f_x = 0$, $2x-3y-8=0 \rightarrow x = \frac{3y}{2}+4$

2nd for $f_y = 0$, $-3x+2y+7=0$; Plug in $x=\frac{3y}{2}+4$,

$$\left(-\frac{9y}{2}-12\right)+2y+7=0 \ ; \ -\frac{5y}{2}-5=0 \ ; \ y=-2$$

At $y=-2$, $x = \frac{3(-2)}{2}+4 = -3+4 = 1 \Rightarrow \boxed{(1,-2) \text{ is critical pt}}$

To classify, need D at $(1,-2)$, $D(1,-2) = f_{xx}(1,-2)f_{yy}(1,-2) - f_{xy}^2(1,-2)$.

$f_{xx}(x,y)=2$, $f_{yy}(x,y)=2$, $f_{xy}(x,y)=-3$. They're constants.

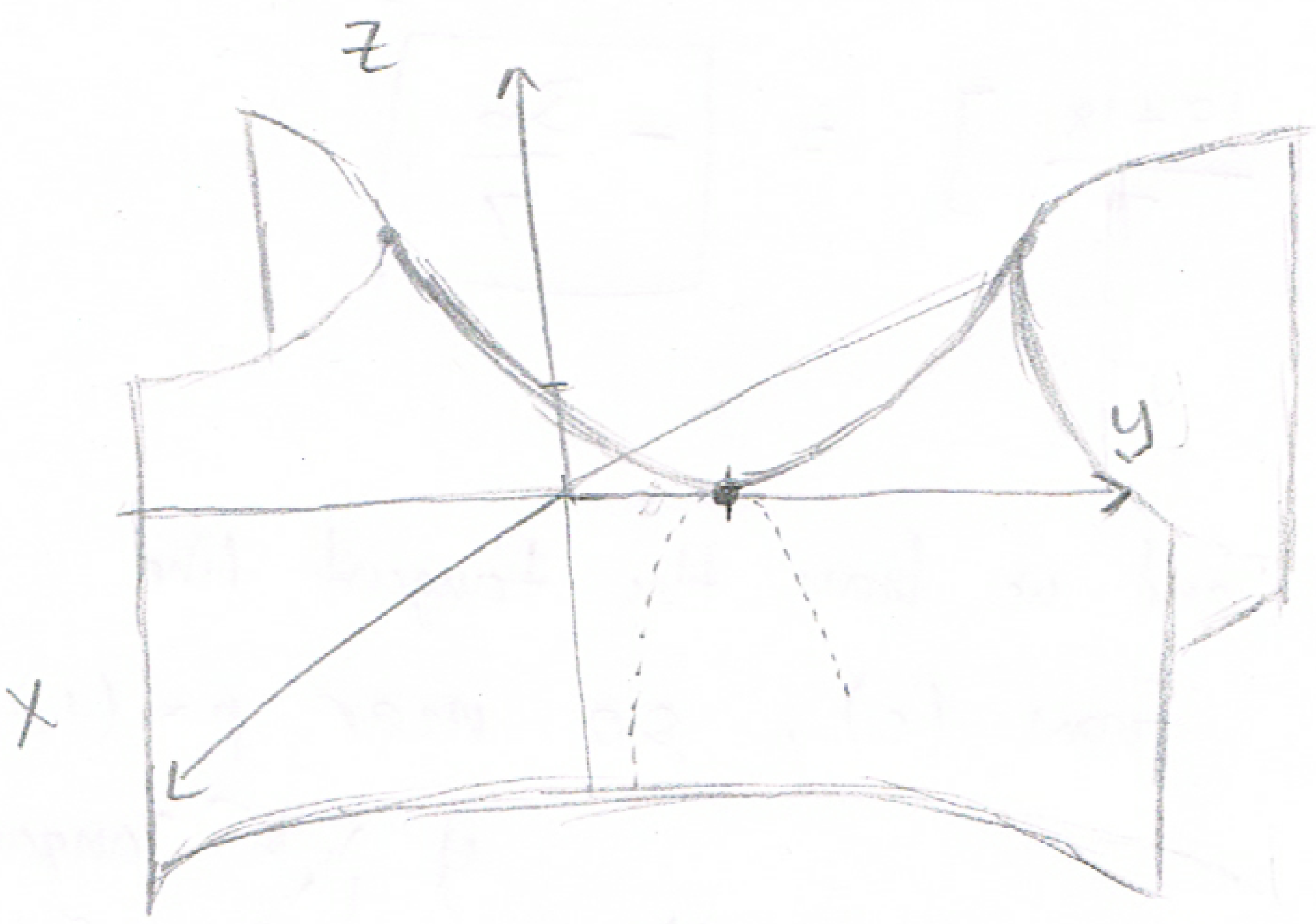
So, $D(1,2) = 2\cdot 2 - (-3)^2 = 4-9 = -5$

since $D(1,2) = -5 < 0$, we have a $\boxed{\text{saddle pt}}$

(so, the point $(1,2)$ is a saddle pt)

b. $z-1 = y^2-x^2-2y \Rightarrow z = y^2-2y+1-x^2 \Rightarrow \boxed{z=(y-1)^2-x^2}$

This is a hyperbolic paraboloid saddled at $(0,1,0)$,



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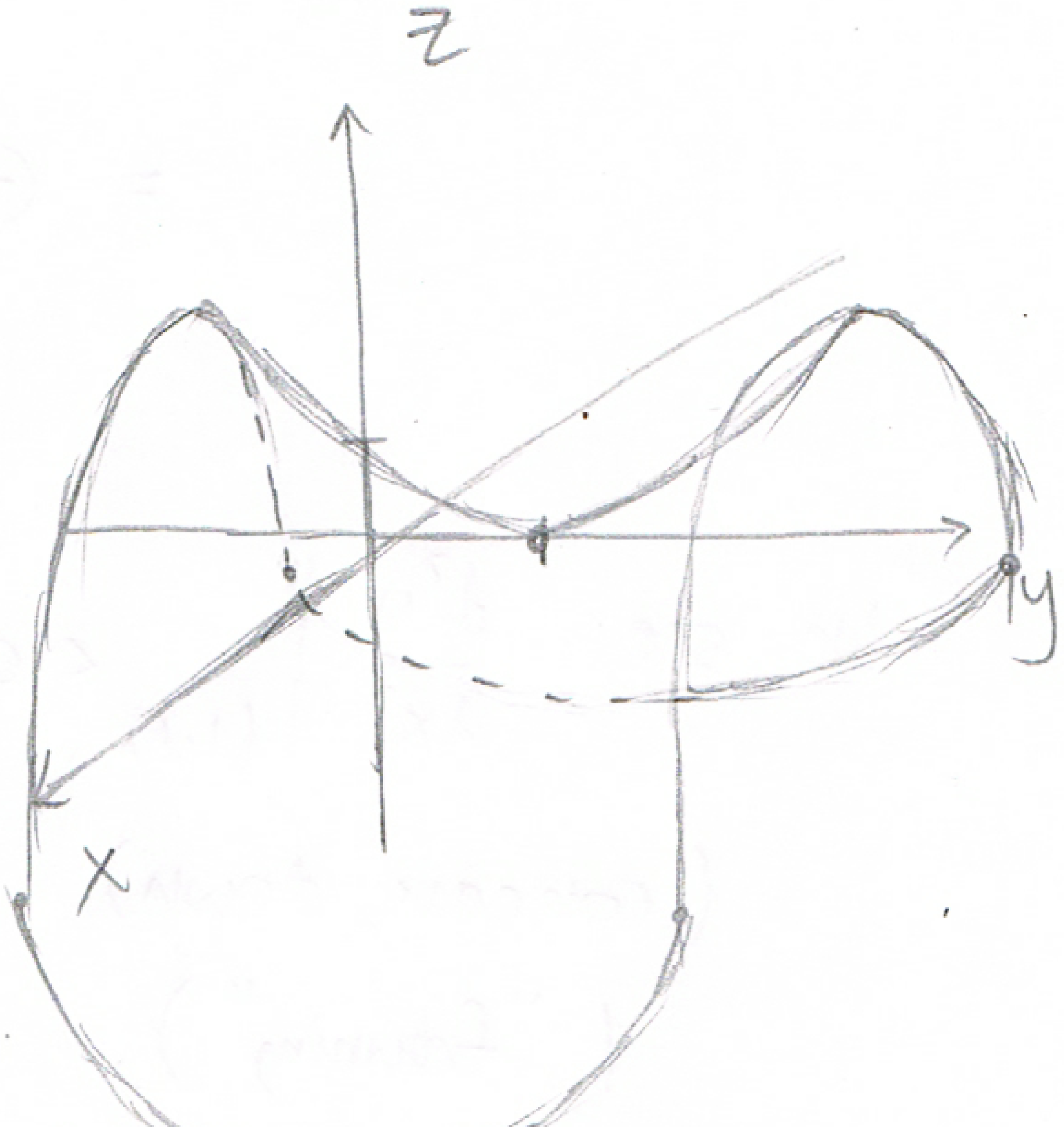

Keys: • when $x=0$, $z=(y-1)^2$, parabola.

• when $y=1$, $z=-x^2$ upside down parabola.

• when $z=k$, hyperbolas.